K: An Alternative Heat Equation Derivation

In the Notes the heat equation is derived in section 3 via a conservation of mass law. There is a rich interpretation of the equation, and its solution, if we outline an alternative derivation of the equation. This is motivated by observations made in 1827 by a famous botanist, Robert Brown, who studied the very animated and irregular motion of small pollen grains in water. The idea here is to start with a discrete process and assume a limiting process exists that converges to a continuous process. The mathematically rigorous details of this convergence is beyond the scope of these Notes, but a more informal approach will give us what we want. The continuous case is *Brownian motion*, but we start with a *random walk* approach. We restrict our argument to the one dimensional lattice (space) case, but generalization to higher dimensions is straight forward.

Consider the real line divided into intervals of length $\triangle x$ and we start at $x = 0$ at $t = 0$. For the next (discrete) time increment $\triangle t$ we flip a coin. If it comes up heads we move to the right $\triangle x$ units. If the coin comes up tails we move to the left $\triangle x$ units. We continue iterating this rule for $t = \triangle t, 2\triangle t, 3\triangle t, \ldots$.Where are we after n moves? We can not say precisely because it is a random process, so we have to give a probabilistic answer. So fix x at some $k\triangle x$ and define $v(x,t)$ to be the probability that after n steps, the random variable X_n is at x; that is, $v(x,t) = Pr\{X_n = x\}$ at time $t = n\triangle t$. We do not need to assume we have a fair coin. Hence, let p (respectively, q) be the probability the coin comes up heads (resp. comes up tails). Thus, $p + q = 1$. Therefore, by the conservation principle, we have

$$v(x, t + \triangle t) = pv(x - \triangle x, t) + qv(x + \triangle x, t) \quad .$$

(We arrive at location x at time $t + \triangle t$ from either $x - \triangle x$ with probability p, or from $x + \triangle x$ with probability q.)

Now we make a couple giant leaps of faith. The first is to assume we can expand v in a Taylor series:

$$v(x, t + \triangle t) = v(x,t) + \triangle t \; v_t(x,t) + \text{terms of order } \triangle t^2 \text{ and higher}$$

$$v(x \pm \triangle x, t) = v(x,t) \pm \triangle x \; v_x(x,t) + \frac{\triangle x^2}{2} \; v_{xx}(x,t) + \text{order } \triangle x^3 \text{ terms and higher}$$

For notational convenience let us write the remainder terms in the $v(x, t + \triangle t)$ expression as $O(\triangle t^2)$, and the remainder terms in the expression for $v(x \pm \triangle x, t)$ as $O(\triangle x^3)$. Then we have

$$v(x,t) + \triangle t\ v_t(x,t) + O(\triangle t^2) = pv(v - \triangle x, t) + qv(x + \triangle x, t)$$

$$= (p+q)v(x,t) + (q-p)\triangle x\ v_x(x,t) + (p+q)\frac{\triangle x^2}{2}\ v_{xx}(x,t) + O(\triangle x^3)$$

or

$$\triangle t\ v_t(x,t) + O(\triangle t^2) = (q-p)\triangle x\ v_x(x,t) + \frac{\triangle x^2}{2}\ v_{xx}(x,t) + O(\triangle x^3)$$

Assuming we are actually flipping a fair coin, so that $p = q = 1/2$, and dividing by $\triangle t$ gives us

$$v_t(x,t) + O(\triangle t) = \frac{\triangle x^2}{2\triangle t}\ v_{xx}(x,t) + O(\triangle x^3/\triangle t)$$

Remark: By the way, the above Taylor expansions are the basis of all finite difference methods for the numerical approximation of solutions to differential equations.

Next bit of faith is that we take finer and finer grids in x, t space by letting $\triangle x \to 0, \triangle t \to 0$, but in such a way that the limit $\lim \frac{\triangle x^2}{2\triangle t} = D > 0$. Hence, in the limit we obtain the heat equation

$$v_t(x,t) = Dv_{xx}(x,t) \tag{1}$$

Remark: If we did not assume a fair coin so that $p \neq q$, then a drift term would appear in equation (1), say

$$v_t(x,t) = -cv_x(x,t) + Dv_{xx}(x,t)$$

Remark: $v(x,t)$ in (1) is now interpreted as a *probability density* associated with the *continuous* random variable x at time t, rather than as a *probability distribution* for the discrete case of a random walk model. Therefore, at time t,

$$Pr\{a \leq x \leq b\} = \int_a^b v(x,t)\ dx \quad .$$

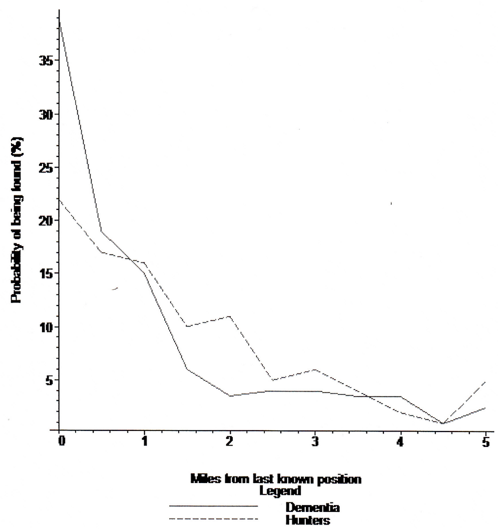

Figure 1: Average distances of lost persons in two categories from the start of the search to where they are found, based on data from Koester, *Lost Person Behavior*, and adapted from Huth's book (see footnote).

The point we are making here is that there is a deep connection between PDEs and probability, one which we do not have time to explore in these Notes. Let us cherry-pick a nontraditional example. Since 2002 there has accumulated a large database of information on people who get lost (very young children, older children, hunters, hikers and backpackers, dementia sufferers, etc.) called the International Search and Rescue Incident Database (ISRID). This has been used to create statistical models of lost people. Figure 1 has been adapted from Huth[1] to illustrate a point. The data tends to fall off in a "Gaussian", that is, a bell-shaped manner. From the Notes we know that a solution to (1) is the fundamental solution $v = S(x,t) = e^{-x^2/4Dt}/\sqrt{4\pi Dt}$, which has such behavior. If one thinks of Dt being estimated to approximate a particular data set like what is pictured in Figure 1, then as a first approximation one can use the heat equation solution with knowledge of the last known location of the lost person, and the time the person has been missing to get an idea of the scope of the search area needed. Children between the ages of 6 and 12, and dementia sufferers, tend to wonder off in approximately

[1]The Lost Art of Finding Our Way, by physicist John Edward Huth, who discusses briefly the behavior of lost people.

a random walk manner. (Of course, with more information, like the age of the person, the terrain, weather, the type of activity the person way engaged in just before being lost, etc. one can hone in on a more sophisticated approach to the search, but that is not our intent here.)

Remark: In Zauderer's *Partial Differential Equations in Applied Mathematics*, he derives a number of classical equations via a random walk approach. This includes biased and correlated random walk models, and one such equation derived is the **telegrapher's equation** that we will discuss in section 7 (but indicated where it comes from through an electrical circuit approach). If you are interested in the connection of random walks and partial differential equations, Zauderer is the source to consult first.